THERMAL CONDUCTIVITY OF A TWO-LAYER WALL FOR A TIME-VARYING HEAT-TRANSFER COEFFICIENT AND AMBIENT TEMPERATURE

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An analytical solution of the problem of the thermal conductivity of a two-layer wall is presented. The solution is obtained by using a special series expansion of the Green's function.

In engineering it is frequently necessary to protect metal structures against excessive heating by cladding them with a thermal shielding material which combines refractory properties with a low thermal conductivity. The nature of the temperature distribution in such structures is shown in Fig. 1 for one-dimensional problems. There is a sharp temperature drop in the layer of thermal insulation. At the metal–cladding interface

$$t_c(0, \tau) = t_M(0, \tau).$$

Here, and subsequently, the subscript M refers to the metal and c to the cladding. The thermal fluxes to the left and right of the boundary are:

$$\lambda_M \left(\frac{\partial t}{\partial x}\right)_M = \lambda_c \left(\frac{\partial t}{\partial x}\right)_c.$$

Since $\lambda_M \gg \lambda_c$ the temperature rise in the metal layer is practically independent of its thermal conductivity and is determined by the heat conducted through the cladding. Therefore one can assume without large error that the temperature is constant through the metal wall.

Under these assumptions, and taking account of the fact that the heat flux from the metal wall into the surrounding medium is negligibly small, the problem of finding the temperature distribution in the two-layer wall is reduced to the integration of the heat-conduction equation

$$a_c \frac{\partial^2 t(x, \tau)}{\partial x^2} = \frac{\partial t(x, \tau)}{\partial \tau} \tag{1}$$

over the range $\langle 0, \delta_c \rangle$ for the following boundary and initial conditions:

$$\lambda_c \frac{\partial t(x, \tau)}{\partial x}\bigg|_{x=0} = c_M \gamma_M \delta_M \frac{\partial t(x, \tau)}{\partial \tau}\bigg|_{x=0},$$

$$\lambda_c \frac{\partial t(x, \tau)}{\partial x}\bigg|_{x=\delta_c} = \alpha(\tau)\left[t_c(\tau) - t(\delta_c, \tau)\right], \tag{2}$$

$$t(x, 0) = t_0. \tag{3}$$

The solution is obtained in two stages. In the first stage the temperature distribution $t_1(x, \tau)$ in the thermal insulation is determined in the absence of the metal wall, i.e., for the boundary conditions:

$$\frac{\partial t_1(x, \tau)}{\partial x}\bigg|_{x=0} = 0,$$

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$$\lambda_c \frac{\partial t_1(x, \tau)}{\partial x}\bigg|_{x=\delta_c} = \alpha(\tau)\left[t_a(\tau) - t_1(\delta_c, \tau)\right]. \tag{4}$$

The solution of this problem is given in [2] and we discuss only certain of its features here.

The system (1)-(4) is reduced to a Fredholm integral equation of the second kind

$$Z(x, \tau) = -\frac{1}{a_c}\int_0^{\delta_c} G(x, s, \tau) \times \left[\frac{\partial z(s, \tau)}{\partial \tau} + \frac{dt_a(\tau)}{d\tau}\right] ds, \tag{5}$$

whose kernel $G(x, s, \tau)$ is the Green's function for a homogeneous differential operator of the second order with homogeneous boundary conditions.

Approximating the kernel of Eq. (5) by a special uniformly convergent bilinear series [1]

$$G(x, s, \tau) = \sum_{i=1}^{n} \gamma_i(x, \tau)\gamma_i(s, \tau), \tag{6}$$

we obtain a system of differential equations with variable coefficients

$$\sum_{k,j=1}^{n} A_{k,j}(\tau)\dot{\varphi}_j(\tau) + B_{k,j}(\tau)\varphi_j(\tau) + \varphi_k(\tau) = \sum_{k=1}^{n} D_k'(\tau), \tag{7}$$

where

$$A_{k,j}(\tau) = \frac{1}{a_c}\int_0^{\delta_c} \gamma_k(s, \tau)\gamma_j(s, \tau)\, ds;$$

$$B_{k,j}(\tau) = \frac{1}{a_c}\int_0^{\delta_c} \gamma_k(s, \tau)\dot{\gamma}_j(s, \tau)\, ds;$$

$$D_k(\tau) = \frac{1}{a_c}\frac{dt_a(\tau)}{d\tau}\int_0^{\delta_c} \gamma_k(s, \tau)\, ds \tag{8}$$

with the initial conditions

$$\varphi_k(0) = \frac{[t_a(0) - t_0]\int_0^{\delta_c} p_k^*(s, 0)\, ds}{\int_0^{\delta_c} \gamma_k(s, 0) p_k^*(s, 0)\, ds}. \tag{9}$$

The system (7) with conditions (9) can be solved by repeated application of the method presented here. For a heat-transfer coefficient in (2) that does not vary with time the system (7) becomes a system of ordinary differential equations with constant coefficients which can easily be solved by familiar methods.

Finally the temperature function for conditions (4) is given by

$$t_1(x, \tau) = t_a(\tau) - \sum_{i=1}^{n} \gamma_i(x, \tau)\varphi_i(\tau). \tag{10}$$

In practical applications the special series (6) representing the kernel of the integral equation (5) in Eq. (10) can be broken off after two terms, and often after the first term.

The second stage of the solution requires the determination of the temperature distribution $t_2(x, \tau)$ in the layer of insulating material in the presence of the metal wall, considered as a plane heat source of strength $q(\tau)$ proportional to the rate of change of temperature at $x = 0$. Then the corresponding function characterizing the temperature distribution in the range $\langle 0, \delta_c \rangle$ is obtained by multiplying the Green's function by $q(\tau)$, i.e.,

$$t_2(x, \tau) = -G(x, 0, \tau)q(\tau) = -\delta_c\left(\frac{\lambda_c}{\alpha(\tau)\delta_c} + 1 - \frac{x}{\delta_c}\right)\frac{c_M\gamma_M\delta_M}{\lambda_c}\frac{\partial t(x, \tau)}{\partial \tau}\bigg|_{x=0}. \tag{11}$$

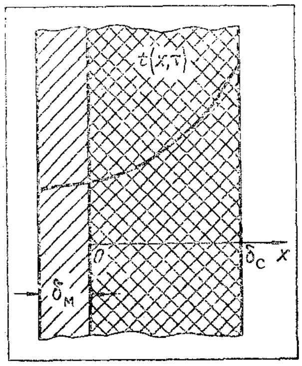

Fig. 1. Temperature distribution in a thermally shielded wall.

Since the problem is linear the required temperature satisfying conditions (2) is

$$t(x,\ \tau)=t_a(\tau)-\sum_{i=1}^{n}\gamma_i(x,\ \tau)\,\varphi_i(\tau)$$
$$-\delta_C\left(\frac{\lambda_C}{\alpha(\tau)\,\delta_C}+1-\frac{x}{\delta_C}\right)\frac{c_M\gamma_M\delta_M}{\lambda_C}\left.\frac{\partial t(x,\ \tau)}{\partial\tau}\right|_{x=0}. \tag{12}$$

Integration of (12) for x = 0 and condition (3) gives

$$t(x,\ \tau)|_{x=0}=\left[t_0+\int_0^{\tau}Q(\tau)\exp\left(\int_0^{\tau}p(\tau)\,d\tau\right)d\tau\right]\exp\left(-\int_0^{\tau}p(\tau)\,d\tau\right), \tag{13}$$

where

$$p(\tau)=\frac{\lambda_C\,\mathrm{Bi}(\tau)}{(1+\mathrm{Bi}(\tau))\,c_M\gamma_M\delta_M\delta_C};$$

$$Q(\tau)=\left[t_a(\tau)-\sum_{i=1}^{n}\gamma_i(0,\ \tau)\,\varphi_i(\tau)\right]p(\tau);$$

$$\mathrm{Bi}(\tau)=\frac{\alpha(\tau)\,\delta_C}{\lambda_C}.$$

Finally the solution of the problem is obtained by substituting (13) into (12)

$$t(x,\ \tau)=t_a(\tau)-\sum_{i=1}^{n}\gamma_i(x,\ \tau)\,\varphi_i(\tau)+\left[1-\frac{x}{\delta_C}\ \frac{\mathrm{Bi}(\tau)}{1+\mathrm{Bi}(\tau)}\right]\left\{t_a(\tau)-\sum_{i=1}^{n}\gamma_i(0,\ \tau)\,\varphi_i(\tau)\right.$$
$$\left.+\left[t_0+\int_0^{\tau}Q(\tau)\exp\left(\int_0^{\tau}p(\tau)\,d\tau\right)d\tau\right]\exp\left(-\int_0^{\tau}p(\tau)\,d\tau\right)\right\}. \tag{14}$$

The running temperature of the metal layer is also given by (14) by setting x = 0.

We present the expression for the temperature function (14), limiting ourselves in (10) to the first term of the series.

In accord with [2] and (14) we have:

$$\gamma_1(x,\ \tau)=\sqrt{\delta_C}\,\frac{\frac{1}{\mathrm{Bi}(\tau)}+\frac{1}{2}\left(1-\frac{x^2}{\delta_C^2}\right)}{\sqrt{\frac{1}{\mathrm{Bi}(\tau)}+\frac{1}{3}}},\quad A_{1,1}(\tau)=\frac{\delta_C^2}{a_C}\left[\frac{1}{\mathrm{Bi}(\tau)}+\frac{1}{3}\right],$$

$$\varphi_1(\tau)=\frac{1}{\sqrt{\left[\frac{1}{\mathrm{Bi}(\tau)}+\frac{1}{3}\right]\delta_C}}\left\{\int_0^{\tau}\frac{dt_a(\tau)}{d\tau}\exp\left(\int_0^{\tau}\frac{d\tau}{A_{1,1}(\tau)}\right)+t_a(0)-t_0\right\}\exp\left(-\int_0^{\tau}\frac{d\tau}{A_{1,1}(\tau)}\right),$$

$$t(x,\ \tau)=t_a(\tau)-\gamma_1(x,\ \tau)\,\varphi_1(\tau)+\left[1-\frac{x}{\delta_C}\ \frac{\mathrm{Bi}(\tau)}{1+\mathrm{Bi}(\tau)}\right]\left\{t_a(\tau)-\gamma_1(0,\ \tau)\,\varphi_1(\tau)\right.$$
$$+\left[t_0+\int_0^{\tau}(t_a(\tau)-\gamma_1(0,\ \tau)\,\varphi_1(\tau))\,p(\tau)\right.$$
$$\left.\left.\exp\left(\int_0^{\tau}p(\tau)\,d\tau\right)d\tau\right]\exp\left(-\int_0^{\tau}p(\tau)\,d\tau\right)\right\}. \tag{15}$$

If the heat-transfer coefficient and the ambient temperature do not vary with time $\alpha(\tau)=\alpha$, $t_a(\tau)=t_a$ and Eq. (15) takes the form

$$t(x,\ \tau)=t_a-(t_a-t_0)\left\{\frac{1+\frac{1}{2}\left(1-\frac{x^2}{\delta_C^2}\right)}{1+\frac{1}{3}\,\mathrm{Bi}}\exp\left(-\frac{\tau}{A_{1,1}}\right)+\left(1-\frac{x}{\delta_C}\ \frac{\mathrm{Bi}}{1+\mathrm{Bi}}\right)\left[\exp(-p\tau)\right.\right.$$

$$
-\frac{1+\frac{1}{2}\mathrm{Bi}}{1+\frac{1}{3}\mathrm{Bi}}\;\frac{A_{1,1}p\exp(-p\tau)-\exp\left(-\frac{\tau}{A_{1,1}}\right)}{A_{1,1}p-1}\Bigg]\Bigg\}. \qquad (16)
$$

Equation (16) can be used to calculate the temperature distribution in the cladding when

$$
\mathrm{Fo}=\frac{a_c}{\delta_c^2}\tau>0.5.
$$

In the metal layer (16) takes the form

$$
t(0,\ \tau)=t_a-(t_a-t_0)\left\{\frac{\frac{2+\mathrm{Bi}}{2(1+\mathrm{Bi})}\left[\exp\left(-\frac{\tau}{A_{1,1}}\right)-\exp(-p\tau)\right]}{\frac{3+\mathrm{Bi}}{3(1+\mathrm{Bi})}-\frac{c_M\gamma_M\delta_M}{c_c\gamma_c\delta_c}}+\exp(-p\tau)\right\}. \qquad (17)
$$

In conclusion we note that the presence of heat sources in the insulating layer does not introduce singularities into the solution of the problem under study.

NOTATION

$t(x,\ \tau)$	is the running temperature;
$a,\ \lambda$	is the thermal diffusivity and conductivity;
δ	is the thickness;
c	is the specific heat;
γ	is the specific weight;
$\alpha(\tau)$, $t_a(\tau)$	are the heat-transfer coefficient and ambient temperature;
τ	is the time.

LITERATURE CITED

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